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Long-Time Behavior of Navier–Stokes Flow on a Two-Dimensional Torus Excited by an External Sinusoidal Force

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In this paper we study the Navier–Stokes flow on the two-dimensional torus $S^1 \times S^1$ excited by the external force $(k^2 \sin ky, 0)$ and find the long-time behavior for the flow starting from some states, where $S^1 = [0, 2\pi](\mathrm{mod}\, 2\pi)$. Especially for the case $k = 2$, it follows from an analysis and computation that the Navier–Stokes flow with the initial state $\cos(mx + ny)$ or $\sin(mx + ny)$ will likely evolve through at most one step bifurcation to either a steady-state solution or a time-dependent periodic solution for any Reynolds number and integers m and n.

KEY WORDS: Navier–Stokes equations; bifurcations; dynamical systems.

1. INTRODUCTION

Let $k \geqslant 2$ be a positive integer and T^2 the two-dimensional torus $S^1 \times S^1$, with S^1 the unit circle $[0, 2\pi)(\mathrm{mod}\, 2\pi)$. We consider an incompressible viscous fluid motion on T^2 sinusoidally excited by an external body force $(k^2 \sin ky, 0)$.

The dynamical behavior of this fluid flow system with k a positive integer defined in terms of velocity $u = (u_1, u_2)$ and pressure p is described by the following Navier–Stokes equations:

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$$
\begin{aligned}
\partial_t u - \Delta u + \lambda u \cdot \nabla u + \nabla p &= (k^2 \sin ky, 0) \\
\nabla \cdot u &= 0 \\
u(t, 0, y) = u(t, 2\pi, y), &\qquad y \in [0, 2\pi), \quad t \geqslant 0 \\
u(t, x, 0) = u(t, x, 2\pi), &\qquad x \in [0, 2\pi), \quad t \geqslant 0
\end{aligned}
\tag{1}
$$

Here Δ and ∇ denote the Laplacian and gradient operators, respectively, $\partial_t = \partial/\partial_t$, and $\lambda > 0$ is the Reynolds number defining the viscous fluid motion.

To ensure the uniqueness of the solution to Eq. (1), we require the additional condition

$$
\int_{T^2} u(x, y)\, dx\, dy = 0 \tag{2}
$$

The problem defined by Eqs. (1) and (2) was first formulated by Kolmogorov[1] and is also referred to as the Kolmogorov problem (see, for example, Okamoto and Shoji[18]).

This fluid motion with $k = 1$ and $\lambda > 0$ is simple, since Meshalkin and Sinai[17] obtained the flow described by the equations

$$
\begin{aligned}
\partial_t u - \Delta u + u \cdot \nabla u + \nabla p &= \lambda(\sin ky, 0) \\
\nabla \cdot u &= 0
\end{aligned}
\tag{3}
$$

when $k = 1$ associated with Eq. (2) on T^2 is attracted by a single steady-state solution for all real λ. This global stability result was also re-proved by Marchioro.[14]

Bifurcation analysis on the stationary Navier–Stokes flow represented by the equations

$$
\begin{aligned}
-\nu\, \Delta u + u \cdot \nabla u + \nabla p &= \gamma(\sin y, 0) \\
\nabla \cdot u &= 0
\end{aligned}
\tag{4}
$$

on the torus $(S^1/a) \times S^1$ with $0 < a < 1$ was examined by Iudovich.[8] It is interesting to note that Okamoto and Shoji[18] provided numerical experiments concluding the absence of secondary step bifurcations in Eqs. (4) by computing an n-mode truncation model for Eqs. (4) with $n \geqslant 200$. Such a result is also partially confirmed by the mathematical analysis and numerical experiments herein.

For lower bound estimates for the Hausdorff dimension of the global attractors described by the above equations, see ref. 2. Recently, Liu[12] gave an extended study on the estimates with respect to Eqs. (2)–(3) by developing the technique of Babin and Vishik,[2] Iudovich,[8] and Meshalkin and Sinai.[17] For the numerical analysis of a Navier–Stokes flow in a bounded domain attracted by a steady state, see refs. 4 and 20. Franceschini *et al.*[5] and Jolly[9] made computational studies on the long-time behavior of the solutions to Eqs. (1)–(2) with more general forcing terms.

Here we consider the dynamical problem from a more analytical viewpoint.

For convenience of analysis, we use the stream function ψ with vorticity $\Delta\psi = \partial_y u_1 - \partial_x u_2$ and rewrite Eqs. (1)–(2) in the form

$$\partial_t \psi - \Delta\psi + \lambda\, \Delta^{-1}(\partial_y \psi\, \partial_x \Delta\psi - \partial_x \psi\, \partial_y \Delta\psi) = -k \cos ky \tag{5}$$

associated with the modified condition

$$\int_{T^2} \psi(x, y)\, dx\, dy = 0 \tag{6}$$

ensuring uniqueness of solution. Thus Eqs. (5)–(6) define an infinite-dimensional dynamical system in the Hilbert space

$$H^2 = \left\{ \psi \in L^2(T^2; R) \,\middle|\, \Delta\psi \in L^2(T^2, R), \int_{T^2} \psi(x, y)\, dx\, dy = 0 \right\}$$

associated with the norm

$$\|\Delta\psi\| = \left(\int_{T^2} |\Delta\psi|^2\, dx\, dy \right)^{1/2}$$

We shall use the concepts with respect to dynamical systems as defined in ref. 7.

By Fourier expansion the solution $\psi(t, x, y)$ can be written as

$$\psi(t, x, y) = \sum_{n=1}^{\infty} \xi_n(t) \cos ny + \sum_{m=1,\, n=-\infty}^{\infty} (\eta_{m,n}(t) \cos(mx + ny) + \zeta_{m,n}(t) \sin(mx + ny))$$

Given a dynamical system starting from an initial state, it is not easy to predict if the system will evolve through a sequence of bifurcations to a steady state or a periodic state or even a fully chaotic state. However, as far as Eqs. (5)–(6) are concerned, it is possible to predict the long-time behavior of this dynamical system starting from some interesting initial states.

In this study, by analysis and computation we discuss the long-time behavior for the solutions to Eqs. (5)–(6) with initial states in either of the following flow invariance subspaces:

$$H^2_{l,k} = \left\{ \psi \in H^2 \,\middle|\, \psi = \sum_{n=1}^{\infty} \xi_n \cos nky + \sum_{m=1,\, n=-\infty}^{\infty} \eta_{m,n} \cos(mlx + nky) \right\}$$

$$\tilde{H}^2_{l,k} = \left\{ \psi \in H^2 \,\middle|\, \psi = \sum_{n=1}^{\infty} \xi_n \cos nky + \sum_{m=1,\, n=-\infty}^{\infty} \eta_{m,n} \sin(2lmx - lx + 2nky) + \sum_{m=1,\, n=-\infty}^{\infty} \zeta_{m,n} \cos(2mlx + 2nky) \right\}$$

where $l \geqslant 0$ is an integer. We find that every solution to Eqs. (5)–(6) with initial state in $H^2_{l,k}$ (resp. $\tilde{H}^2_{l,k}$) will likely go through at most one step pitchfork bifurcation and evolve toward a steady-state solution in $H^2_{l,k}$ (resp. $\tilde{H}^2_{l,k}$).

Additionally, in order to give more evidence for the long-time behavior of other solutions to Eqs. (5)–(6), we take $k = 2$ as an example to study the fluid motion in either of the following flow invariance subspaces:

$$\mathscr{H}^2_{l,2} = \left\{ \psi \in H^2 \,\middle|\, \psi = \sum_{n=1}^{\infty} \xi_n \cos 2ny + \sum_{m=1,\, n=-\infty}^{\infty} \eta_{m,n} \cos(2mlx - lx + 2ny + y) + \sum_{m=1,\, n=-\infty}^{\infty} \zeta_{m,n} \cos(2mlx + 2ny) \right\}$$

$$\tilde{\mathscr{H}}^2_{l,2} = \left\{ \psi \in H \,\middle|\, \psi = \sum_{n=1}^{\infty} \xi_n \cos 2ny + \sum_{m=1,\, n=-\infty}^{\infty} \eta_{m,n} \sin(2mlx - lx + 2ny + y) + \sum_{m=1,\, n=-\infty}^{\infty} \zeta_{m,n} \cos(2mlx + 2ny) \right\}$$

for integer $l \geqslant 0$. Our investigation shows that the attractor of Eqs. (5)–(6) with $k = 2$ reduced in either $\mathscr{H}^2_{l,2}$ or $\tilde{\mathscr{H}}^2_{l,2}$ for $l \neq 1$ coincides with the steady-state solution $-(1/2) \cos 2y$. Moreover, when the Reynolds number varies, $-(1/2)\cos 2y$ loses stability in $\mathscr{H}^2_{1,2}$ (resp. $\tilde{\mathscr{H}}^2_{1,2}$), and bifurcates into a time-dependent periodic solution which is stable in $\mathscr{H}^2_{1,2}$ (resp. $\tilde{\mathscr{H}}^2_{1,2}$).

Consequently, for any Reynolds number λ, integers m and n, and for any solution $\psi(t, x, y)$ to (5)–(6) with $k=2$ starting from either $\psi(0, x, y) = \cos(mx + ny)$ or $\psi(0, x, y) = \sin(mx + ny)$ in H^2, it seems that $\psi(t, x, y)$ approaches either a steady-state solution or a time-dependent periodic solution.

In another study, we shall, however, find that Eqs. (5)–(6) with $k \geqslant 3$ have a time-dependent periodic solution which undergoes further step bifurcations other than pitchfork and Hopf bifurcations as the Reynolds number λ increases.

The outline of this paper is as follows: In Section 2, we investigate pitchfork bifurcation for Eqs. (5)–(6) reduced in $H^2_{l,k}$. In Section 3, we provide a four-mode truncation model for Eqs. (5)–(6) reduced in $H^2_{l,k}$ and show that, for this truncated system, every bifurcated equilibrium point is always stable irrespective of the magnitude of the Reynolds number. In Section 4, to support this stable criterion, we additionally introduce a 17-mode truncation model for Eqs. (5)–(6) reduced in $H^2_{l,k}$. Numerical experiments on this model with $k=2$ and 3 corroborates the findings in Sections 3 and 4. In Section 5, we examine time-dependent periodic solutions of Eqs. (5)–(6) with $k=2$ reduced in $\mathscr{H}^2_{l,2}$. Finally, in Section 6, we present some remarks showing that the results in Sections 3–5 remain valid whenever $H^2_{l,k}$ and $\mathscr{H}^2_{l,2}$ are respectively replaced by $\tilde{H}^2_{l,k}$ and $\tilde{\mathscr{H}}^2_{l,2}$.

2. PITCHFORK BIFURCATION

In this section we study pitchfork bifurcation for Eqs. (5)–(6) in $H^2_{l,k}$.

Let us first note that $H^2_{l,k} \supset H^2_{2l,k}$ and every subspace $H^2_{l,k}$ is flow invariant with respect to Eqs. (5)–(6).

Lemma 2.1. Let $l \geqslant 0$ and $\lambda > 0$. Then for every initial function $\psi_0 \in H^2_{l,k}$, Eqs. (5)–(6) admit a unique global solution

$$\psi \in C([0, \infty); H^2_{l,k})$$

Proof. For every $j \geqslant 1$, it is not difficult to verify that

$$\Delta^{-1}(\partial_y \psi_j \partial_x \Delta\psi_j - \partial_x \psi_j \partial_y \Delta\psi_j) \in H^2_{l,k}$$

with

$$\psi_j = \sum_{n=1}^{j} \xi_n \cos nky + \sum_{m=1,\, n=-j}^{j} \eta_{m,n} \cos(mlx + nky)$$

By an elementary manipulation, the desired assertion follows from the Galerkin approximation procedure (see, for example, ref. 19). The proof is complete.

With the use of Lemma 2.1, we rewrite Eqs. (5)–(6) in the form of a functional ordinary differential equation

$$\frac{d\psi}{dt} - \Delta\psi + B_{k,\lambda}(\psi) = 0, \qquad \psi(t)(\cdot,\cdot) = \psi(t,\cdot,\cdot) \in H^2_{1,k} \tag{7}$$

representing an infinite-dimensional dynamical system, where

$$B_{k,\lambda}(\psi) = \lambda\Delta^{-1}(\partial_y\psi\,\partial_x\Delta\psi - \partial_x\psi\,\partial_y\Delta\psi) + k\cos ky$$

since the pitchfork bifurcation problem for this system is largely based on the spectral behavior of the operator

$$-\Delta + \lambda A_k = -\Delta + \lambda\Delta^{-1}\sin ky(\Delta + k^2)\,\partial_x$$

the linearized operator of the stationary Navier–Stokes equation $-\Delta\psi + B_{k,\lambda}(\psi) = 0$ at the steady-state solution $-(1/k)\cos ky$, or the Fréchet derivative of the operator $-\Delta\psi + B_{k,\lambda}(\psi)$ at $-(1/k)\cos ky$.

Let us begin with the investigation of the spectral behavior of the operator $-\Delta + \lambda A_k$ in $H^2_{1,k}$. This spectral problem was partially studied in H^2 by Iudovich[8] and later by Liu[12] in a similar way. However, this problem is now investigated in $H^2_{1,k}$ together with its subspaces. It is convenient to prove the following lemmas in detail by an approach developed from the technique proposed in refs. 8, 13, and 17. Also see refs. 4 and 20 for the study of the resolvent estimates of the linearized Navier–Stokes operator in a general bounded domain.

Lemma 2.2. For $k \geqslant 2$, there exist exactly $k-1$ real functions and $k-1$ real numbers

$$\rho_1(\lambda) < \cdots < \rho_{k-1}(\lambda) \quad \text{with} \quad \lambda > 0 \quad \text{and} \quad 0 < \lambda_1 < \cdots < \lambda_{k-1}$$

satisfying

$$\rho_l(\lambda_l) = 0, \qquad \frac{2(k^2+l^2)^2}{k^2-l^2} < \lambda_l^2, \qquad \frac{d}{d\lambda}\rho_l(\lambda) < 0$$

$$\dim\ker(-\Delta + \lambda A_k - \rho) \leqslant 1 \qquad \text{in } H^2_{1,k}$$

for $l = 1,..., k-1$ and $\rho \in \{\rho \in C \mid \operatorname{Re}\rho < k^2\}$. This equality holds if and only if $\rho = \rho_l(\lambda)$.

Proof. First, to study this spectral problem, we can suppose, without loss of generality, that H^2 is a complex space. For every $l \geqslant 0$ we see that

$$\left\{\psi \in H^2 \,\middle|\, \psi = \sum_{m=-\infty}^{\infty} \xi_m \cos(lx + mky)\right\}$$

is an invariant subspace with respect to the operator $-\Delta + \lambda A_k$; the spectral problem

$$-\Delta\psi + \lambda A_k \psi - \rho\psi = 0$$

can be reduced in this subspace instead of in $H^2_{1,k}$. Thus by an elementary manipulation, this spectral problem with an eigenfunction

$$\psi = \sum_{m=-\infty}^{\infty} \xi_{l,m} \cos(lx + mky) \qquad \text{in } H^2 \tag{8}$$

becomes

$$\frac{\lambda\alpha_{l,m-1}\xi_{l,m-1}}{2\beta_{l,m}} - \frac{\lambda\alpha_{l,m+1}\xi_{l,m+1}}{2\beta_{l,m}} + (\beta_{l,m} - \rho)\,\xi_{l,m} = 0 \tag{9}$$

for any integer m, where

$$\beta_{l,m} = l^2 + m^2k^2 \qquad \text{and} \qquad \alpha_{l,m} = l(l^2 + m^2k^2 - k^2) \tag{10}$$

It is not difficult to find that $\xi_{l,m} \neq 0$ and $\xi_{0,m} = 0$ for any integers $l \geqslant 1$ and m. This allows us to define, for $m \geqslant 0$ and $l \geqslant 1$,

$$\gamma_{l,m} = \frac{\alpha_{l,m}\xi_{l,m}}{\alpha_{l,m-1}\xi_{l,m-1}} \qquad \text{and} \qquad \gamma_{l,-m} = \frac{\alpha_{l,-m}\xi_{l,-m}}{\alpha_{l,1-m}\xi_{l,1-m}} \tag{11}$$

In particular, we define $1/\gamma_{k,\pm 1} = 0$ since $\alpha_{k,0} = 0$. Thus we see that Eq. (9) is equivalent to the following system of algebraic equations:

$$\frac{2(\beta_{l,0} - \rho)\,\beta_{l,0}}{\lambda\alpha_{l,0}} + \gamma_{l,-1} = \gamma_{l,1} \qquad \text{when} \quad l \neq k$$

$$\frac{2(\beta_{l,m} - \rho)\,\beta_{l,m}}{\lambda\alpha_{l,m}} + \frac{1}{\gamma_{l,m}} = \gamma_{l,m+1} \qquad \text{for} \quad m \geqslant 1$$

$$\frac{2(\beta_{l,m} - \rho)\,\beta_{l,m}}{\lambda\alpha_{l,m}} + \gamma_{l,m-1} = \frac{1}{\gamma_{l,m}} \qquad \text{for} \quad m \leqslant -1$$

Since $\psi \in H^2$ implies the boundedness of $\gamma_{l,\pm m}$ with respect to m, we have $\gamma_{l,\pm m} \to 0$ as $m \to \pm\infty$, and hence

$$\gamma_{l,m} = -\gamma_{l,-m} = \cfrac{-1}{\cfrac{2(\beta_{l,m}-\rho)\,\beta_{l,m}}{\lambda\alpha_{l,m}} + \cfrac{1}{\cfrac{2(\beta_{l,m+1}-\rho)\,\beta_{l,m+1}}{\lambda\alpha_{l,m+1}} + \cfrac{1}{\ddots}}}, \qquad m \geqslant 1 \tag{12}$$

and so

$$\frac{2(\beta_{l,1}-\rho)\,\beta_{l,1}}{\lambda\alpha_{l,1}} = \cfrac{-1}{\cfrac{2(\beta_{l,2}-\rho)\,\beta_{l,2}}{\lambda\alpha_{l,2}} + \cfrac{1}{\cfrac{2(\beta_{l,3}-\rho)\,\beta_{l,3}}{\lambda\alpha_{l,3}} + \cfrac{1}{\ddots}}}, \qquad l = k \tag{13}$$

$$\frac{(\beta_{l,0}-\rho)\,\beta_{l,0}}{\lambda\alpha_{l,0}} = \cfrac{-1}{\cfrac{2(\beta_{l,1}-\rho)\,\beta_{l,1}}{\lambda\alpha_{l,1}} + \cfrac{1}{\cfrac{2(\beta_{l,2}-\rho)\,\beta_{l,2}}{\lambda\alpha_{l,2}} + \cfrac{1}{\ddots}}}, \qquad l \neq k \tag{14}$$

It is readily seen that Eqs. (13)–(14) are not true whenever $l \geqslant k$ and $\operatorname{Re}\rho \leqslant 4$. Therefore it is sufficient to examine Eq. (14) with $l = 1,\dots, k-1$.

Second, to show the lack of a nonreal eigenvalue ρ, we suppose, on the contrary, $\operatorname{Im}\rho \neq 0$. Then, for $m \geqslant 0$,

$$\left|\arg\left(\frac{2(\beta_{l,m}-\rho)\,\beta_{l,m}}{\lambda\alpha_{l,m}}\right)\right| = |\arg(\beta_{l,m}-\rho)| > \left|\arg\left(\frac{2(\beta_{l,m+1}-\rho)\,\beta_{l,m+1}}{\lambda\alpha_{l,m+1}}\right)\right|$$

This together with Eq. (14) yields

$$|\arg(\beta_{l,0}-\rho)| = \left|\arg\left(\frac{(\beta_{l,0}-\rho)\,\beta_{l,0}}{\lambda\alpha_{l,0}}\right)\right| < \left|\arg\left(\frac{2(\beta_{l,1}-\rho)\,\beta_{l,1}}{\lambda\alpha_{l,1}}\right)\right| < |\arg(\beta_{l,0}-\rho)|$$

which leads to a contradiction.

Third, to give the functions $\rho_l(\lambda)$, we define, for $m \geqslant 1$,

$$g_{2m-1}(l, \lambda, \rho) = -\frac{2\beta_{l,0}\beta_{l,2m-1}(\beta_{l,0}-\rho)(\beta_{l,2m-1}-\rho)}{\lambda^2\alpha_{l,0}\alpha_{l,2m-1}}$$

$$g_{2m}(l, \rho) = -\frac{2\alpha_{l,0}\beta_{l,2m}(\beta_{l,2m}-\rho)}{\beta_{l,0}\alpha_{l,2m}(\beta_{l,0}-\rho)}$$

Multiplying Eq. (14) by $\lambda\alpha_{l,0}/[(\beta_{l,0}-\rho)\beta_{l,0}]$ yields

$$1 = \cfrac{1}{g_1(l, \lambda, \rho) + \cfrac{1}{g_2(l, \rho) + \cfrac{1}{g_3(l, \lambda, \rho) + \cfrac{1}{g_4(l, \rho) + \ddots}}}}$$

Denoting the right-hand side by $f(l, \lambda, \rho)$ and observing that

$$\partial_\rho g_{2m-1}(l, \lambda, \rho) < 0 < \partial_\rho g_{2m}(l, \rho) \qquad \text{for} \quad \rho < \beta_{l,0}$$

we obtain

$$\partial_\rho f(l, \lambda, \rho) > 0 \qquad \text{for} \quad \rho < \beta_{l,0} \tag{15}$$

Hence the observation

$$\lim_{\rho \nearrow \beta_{l,0}} f(l, \lambda, \rho) = \infty, \qquad \lim_{\rho \to -\infty} f(l, \lambda, \rho) = 0$$

implies the uniqueness and existence of $\rho = \rho_l(\lambda) < \beta_{l,0}$ satisfying

$$1 = f(l, \lambda, \rho_l(\lambda)) \tag{16}$$

For such an eigenvalue ρ_l, it follows from Eq. (11) that

$$\xi_{l,0} = c, \qquad \xi_{l,\pm m} = c\,\frac{\gamma_{l,\pm m} \cdots \gamma_{l,\pm 1}\alpha_{l,0}}{\alpha_{l,\pm m}}, \qquad m \geqslant 1$$

where $c \in R$ is an arbitrary constant; that is, the eigenfunctions with respect to ρ_l form a one-dimensional space.

Finally, it remains to verify the existence of λ_l and the monotonicity of ρ_l and λ_l. From the inequality $\partial_\lambda g_{2m-1}(l, \lambda, \rho) < 0$ it follows that

$$\partial_\lambda f(l, \lambda, \rho) > 0 \qquad \text{for} \quad \rho < \beta_{l,0} \tag{17}$$

This together with

$$\lim_{\lambda \to 0} f(l, \lambda, 0) = 0 \qquad \text{and} \qquad \lim_{\lambda \to \infty} f(l, \lambda, 0) = \infty$$

implies that there is a unique positive number λ_l such that $1 = f(l, \lambda_l, 0)$, which also gives $\lambda_l^2 > 2(k^2 + l^2)^2/(k^2 - l^2)$, since $f(l, \lambda_l, 0) < 1/g_1(l, \lambda_l, 0)$.

To prove the monotonicity of λ_l and ρ_l, we note that, for $1 < l+1 < k$,

$$g_{2m-1}(l+1, \lambda, \rho) > g_{2m-1}(l, \lambda, \rho), \qquad g_{2m}(l+1, \lambda, \rho) < g_{2m}(l, \lambda, \rho)$$

and hence $f(l+1, \lambda, \rho) < f(l, \lambda, \rho)$, which shows that

$$1 = f(l, \lambda_l, 0) = f(l+1, \lambda_{l+1}, 0) < f(l, \lambda_{l+1}, 0)$$
$$1 = f(l, \lambda, \rho_l(\lambda)) = f(l+1, \lambda, \rho_{l+1}(\lambda)) < f(l, \lambda, \rho_{l+1}(\lambda))$$

Consequently, by Eqs. (15) and (17), we have

$$\lambda_l < \lambda_{l+1} \qquad \text{and} \qquad \rho_l(\lambda) < \rho_{l+1}(\lambda) \qquad \text{whenever} \quad 1 \leqslant l \leqslant k-2$$

Furthermore, Eq. (16) gives

$$0 = \frac{df(l, \lambda, \rho_l(\lambda))}{d\lambda} = \partial_\lambda f(l, \lambda, \rho_l(\lambda)) + \partial_\rho f(l, \lambda, \rho_l(\lambda)) \frac{d\rho_l(\lambda)}{d\lambda}$$

which together with Eqs. (15) and (17) implies $(d/d\lambda)\, \rho_l(\lambda) < 0$. The proof is complete.

The spectral problem in Lemma 2.2 may also be considered in $H^2_{l,k}$, the subspaces of $H^2_{1,k}$. In fact, the proof of Lemma 2.2 provides the following result:

Corollary 2.1. For $l = 1,..., k-1$, let $n = [k/l]$ when $[k/l] < k/l$, and let $n = [k/l] - 1$ when $[k/l] = k/l$, where $[k/l]$ denotes the integer part of k/l. Suppose that $\rho_i(\lambda)$ with $i = l, 2l,..., nl$ are defined as in Lemma 2.2. Then, for $\lambda > 0$ and $\rho \in \{\rho \in C \mid \operatorname{Re} \rho < k^2\}$,

$$\dim \ker(-\Delta + \lambda A_k - \rho) \leqslant 1 \qquad \text{in } H^2_{l,k}$$

and the equality holds if and only if $\rho = \rho_i(\lambda)$.

Additionally, the first step in the proof of Lemma 2.2 implies the following result with respect to $H^2_{l,k}$ for either $l=0$ or $l \geqslant k$:

Corollary 2.2. Consider Eq. (7) in the case $l=0$ or $l \geqslant k$. Then the steady-state solution $-(1/k)\cos ky$ is always stable for all $\lambda > 0$.

To obtain a pitchfork bifurcation result, it is necessary to introduce the following simple estimates.

Lemma 2.3. Let $l=1,\dots,k-1$, $\lambda>0$, and $\Sigma_{l,\lambda}$ denote the set of the steady-state solutions of Eq. (7). Then following assertions are valid.

(i) Both $\Delta^{-1}A$ and $\Delta^{-1}B_{k,\lambda}$ are compact and continuous operators mapping $H^2_{l,k}$ into itself, and

$$\|\Delta^{-1}(\partial_y\psi\,\partial_x\Delta\psi - \partial_x\psi\,\partial_y\Delta\psi)\| = o(\|\Delta\psi\|) \qquad \text{for} \quad \psi \in H^2_{l,k}$$

(ii) We have

$$\|\Delta\psi\| \leqslant 2k^2\pi \qquad \text{for} \quad \psi \in \Sigma_{l,\lambda}$$

Proof. (i) We see, for $\psi \in H^2_{l,k}$,

$$\begin{aligned} \|\Delta^{3/2}(\Delta^{-1}A\psi)\| &= \|(-\Delta)^{-1/2}\sin ky\,\partial_x(\Delta+k^2)\,\psi\| \\ &\leqslant \|\sin ky\,(\Delta\psi + k^2\psi)\| \\ &\leqslant \|\Delta\psi\| + k^2\,\|\psi\| \leqslant (1+k^2)\,\|\Delta\psi\| \end{aligned}$$

On the other hand, applying the Sobolev imbedding theorem (see, for example, ref. 6) and the Hölder inequality, we obtain

$$\begin{aligned} &\frac{1}{\lambda}\left\|\Delta^{5/4}\left(\Delta^{-1}B_{k,\lambda}(\psi) + \frac{1}{k}\cos ky\right)\right\| \\ &\quad = \|\Delta^{1/4}\Delta^{-1}(\partial_y\psi\,\partial_x\Delta\psi - \partial_x\psi\,\partial_y\Delta\psi)\| \\ &\quad = \|(-\Delta)^{-3/4}(\partial_x(\partial_y\psi\,\Delta\psi) - \partial_y(\partial_x\psi\,\Delta\psi))\| \\ &\quad \leqslant \|(-\Delta)^{-1/4}(\partial_y\psi\,\Delta\psi)\| + \|(-\Delta)^{-1/4}(\partial_x\psi\,\Delta\psi)\| \\ &\quad \leqslant c_1(\|\partial_y\psi\,\Delta\psi\|_{L^{4/3}} + \|\partial_x\psi\,\Delta\psi\|_{L^{4/3}}) \\ &\quad \leqslant c_1(\|\partial_y\psi\|_{L^4} + \|\partial_x\psi\|_{L^4})\,\|\Delta\psi\| \\ &\quad \leqslant c_2\,\|\Delta\psi\|^2 \end{aligned}$$

where c_1 and c_2 are constants. The continuity of $\Delta^{-1}A$ and $\Delta^{-1}B_\lambda$ is now obvious. By the Kondrachov theorem (see, for example, ref. 6) we see that the norms $\|\Delta^{3/2}\cdot\|$ and $\|\Delta^{5/4}\cdot\|$ are compact with respect to the norm $\|\Delta\cdot\|$, and thus assertion (i) is valid.

(ii) Multiplying the stationary equation with respect to Eq. (7) by $\Delta^2\psi$ and integrating over the torus T^2, we obtain, after integration by parts, immediately the desired estimate

$$\|\Delta\psi\| \leqslant \|\nabla\Delta\psi\| \leqslant \|k^2\cos ky\| \leqslant 2k^2\pi \qquad \text{for} \quad \psi\in\Sigma_{l,\lambda}$$

The proof is complete.

The main result of this section reads as follows:

Theorem 2.1. Let $\lambda_1,\dots,\lambda_{k-1}$ be defined as given in Lemma 2.2, and let $\varphi_0(y)=$ the steady-state solution $-(1/k)\cos ky$. Then the equation $(d/dt)\psi-\Delta\psi+B_{k,\lambda}(\psi)=0$ in $H^2_{1,k}$, Eq. (7), admits $k-1$ supercritical pitchfork bifurcation points $(\lambda_1,\varphi_0),\dots,(\lambda_{k-1},\varphi_0)$; in other words, there exists a small number $\delta>0$ such that, for every $l=1,\dots,k-1$ and every $\lambda_l<\lambda<\lambda_l+\delta$, this equation has two stable equilibrium solutions $\psi_{l,\lambda}$ and $\phi_{l,\lambda}$ satisfying $\psi_{l,\lambda_l}=\phi_{l,\lambda_l}=\varphi_0$. Moreover, this equation has no other bifurcation points along the half-line $\{(\lambda,\varphi_0)\mid\lambda>0\}$.

As a consequence of this theorem and its proof, we can obtain $k-1$ bifurcation points of the stationary system $-\Delta\psi+B_{k,\lambda}(\psi)=0$ in $H^2_{1,k}$ by applying Krasnosel'skii's Theorem.[10] But to deduce the pitchfork bifurcation of Eq. (7), we will provide a detailed proof. This is achieved by using the Larey–Schauder degree method of studying nonuniqueness problems (see, for example, refs. 10 and 19).

Proof. In order to use the Larey–Schauder degree method in the space $H^2_{l,k}$, let us adopt the following notation:

$$\lambda_k=\lambda_{k-1}+1, \qquad r=2k^2\pi, \qquad \delta_0=(\lambda_l+\lambda_{l+1})/2, \qquad l=1,\dots,k-1$$

$$F_\lambda(\psi)=\psi-\lambda\Delta^{-1}A\psi-\Delta^{-1}B_{k,\lambda}(\psi)-\frac{1}{k}\cos ky, \qquad \psi\in H^2_{l,k}$$

$$\Theta_{l,s}=\{\psi\in H^2_{l,k}\mid\|\Delta\psi\|<s\}$$

$$\Omega_{l,r,\varepsilon}=\{\psi\in H^2_{l,k}\mid\varepsilon<\|\Delta\psi\|<r\}, \qquad 0<\varepsilon<r$$

where λ_l is defined in Lemma 2.2.

Now the equilibrium or steady-state solutions of Eq. (7) become the solutions of the equation $F_\lambda(\psi)=0$, which are to be found in the domain $\Omega_{l,r,\varepsilon}$ in the following.

By Lemma 2.3, the choice of r implies

$$\Theta_{l,r} \supset \bigcup_{\lambda \in [\lambda_l, \lambda_l + \delta_0]} \Sigma_{l,\lambda}$$

Using Corollary 2.1, we see that for every $0 < \delta < \delta_0$ there is a small $\varepsilon = \varepsilon(\delta)$ so that the operator $F_\lambda(\psi) = 0$ has only the trivial solution 0 in the ball $\Theta_{l,2\varepsilon}$ for $\lambda_l + \delta < \lambda < \lambda_l + \delta_0$. In view of Lemma 2.3, $\Delta^{-1}A$ and $\Delta^{-1}B_{k,\lambda}$ are compact in $H^2_{l,k}$. Hence, the following Larey–Schauder degrees of F_λ are well defined over $\Theta_{l,r}$, $\Theta_{l,\varepsilon}$, and $\Omega_{l,r,\varepsilon}$ with respect to 0:

$$\deg(F_\lambda, \Theta_{l,r}, 0) \qquad \text{for} \quad 0 < \lambda$$

$$\deg(F_\lambda, \Theta_{l,\varepsilon}, 0),\ \deg(F_\lambda, \Omega_{l,r,\varepsilon}, 0) \qquad \text{for} \quad \lambda_l + \delta < \lambda < \lambda_l + \delta_0$$

On the other hand, for $0 < \lambda < \lambda_l + \delta_0$, if $\sigma_l(\lambda)$ is the eigenvalue of $I - \lambda\Delta^{-1}A$ in $H^2_{l,k}$, or

$$\Delta\psi + \frac{\lambda}{1 - \sigma_l(\lambda)} A\psi = 0 \qquad \text{in} \quad H^2_{l,k}$$

then Corollary 2.1 gives

$$\sigma_l(\lambda) = 1 - \lambda/\lambda_l \qquad \text{for} \quad 0 < \lambda < \lambda_l + \delta_0 \tag{18}$$

This shows, in conjunction with Corollary 2.1, that

$$\deg(F_\lambda, \Theta_{l,r}, 0) = 1, \qquad 0 < \lambda < \lambda_l$$

and so

$$\deg(F_\lambda, \Theta_{l,r}, 0) = 1, \qquad 0 < \lambda < \lambda_l + \delta_0$$

since the choice of r implies that the problem $F_\lambda(\psi) = 0$ has no solutions outside the ball $\Theta_{l,r}$ for $0 < \lambda < \lambda_l + \delta_0$. Applying Corollary 2.1, Lemma 2.3, and Eq. (18), we deduce that

$$\deg(F_\lambda, \Theta_{l,\varepsilon}, 0) = \deg(I - \lambda\Delta^{-1}A, \Theta_{l,\varepsilon}, 0) = -1, \qquad \lambda_l + \delta < \lambda < \lambda_l + \delta_0$$

and

$$\deg(F_\lambda, \Omega_{l,r,\varepsilon}, 0) = \deg(F_\lambda, \Theta_{l,r}, 0) - \deg(F_\lambda, \Theta_{l,\varepsilon}, 0) = 2$$

for $\lambda_l + \delta < \lambda < \lambda_l + \delta_0$. This shows that $F_\lambda(\psi) = 0$ has two solutions in the domain $\Omega_{l,r,\varepsilon}$.

Additionally, by Corollary 2.1, the system $(d/dt)\,\psi - \Delta F_\lambda(\psi) = 0$ in $H^2_{l,k}$ is globally stable and has a unique equilibrium solution 0 when $0 < \lambda < \lambda_l$, and the trivial equilibrium solution 0 possesses a one-dimensional unstable manifold when $\lambda_l < \lambda < \lambda_{l+1}$. Hence 0 can only bifurcate two nontrivial equilibrium solutions, which are stable because of continuity. This implies that $(d/dt)\,\psi - \Delta F_\lambda(\psi) = 0$ in $H^2_{l,k}$ admits exactly two nontrivial stable equilibrium solutions for $\lambda_l + \delta < \lambda < \lambda_l + 2\delta$, provided that $\delta > 0$ is sufficiently small. The proof is complete.

3. FOUR-MODE TRUNCATION SCHEME

The proof of Theorem 2.1 provides the long-time behavior of the infinite-dimensional dynamical system described by

$$(d/dt)\,\psi - \lambda\Delta\psi + B_\lambda(\psi) = 0 \tag{19}$$

in $H^2_{l,k}$ for $l = 1,..., k-1$ when λ is near λ_l. In order to derive more information for all $\lambda > 0$, we truncate Eq. (19) in $H^2_{1,k}$ into $k-1$ systems of ordinary differential equations.

The analysis of Section 2 allows a suitable subspace to be defined on which Eq. (19) in $H^2_{1,k}$ can be projected. Note that the solution of Eq. (19) in $H^2_{1,k}$ is of the form

$$\psi = \sum_{m=1}^{\infty} \xi_m(t)\cos mky + \left(\sum_{l=1}^{k-1}\sum_{m=-\infty}^{\infty} + \sum_{l=k}^{\infty}\sum_{m=-\infty}^{\infty}\right)\xi_{l,m}(t)\cos(lx + mky)$$

Lemma 2.2 shows that the only term contributing to the spectral problem of the operator $-\Delta + \lambda A_k$ is the second one on the right-hand side of this equation. It follows from Lemma 2.2 together with Eqs. (8), (11), and (12) that $-\Delta + \lambda A_k$ in $H^2_{1,k}$ has exactly $k-1$ eigenvalues $\rho_1(\lambda),..., \rho_{k-1}(\lambda)$ associated with $k-1$ eigenfunctions

$$\sum_{m=-\infty}^{\infty} \xi_{l,m}\cos(lx + mky), \qquad l = 1,..., k-1$$

where

$$\gamma_{l,0} = 1, \qquad \xi_{l,\pm m} = \frac{\gamma_{l,\pm m}\cdots\gamma_{l,\pm 1}\alpha_{l,0}}{\alpha_{l,\pm m}}, \qquad m \geqslant 1$$

From Eqs. (10) and (12) we see that

$$|\xi_{l,m}| < -\frac{\lambda^m \alpha_{l,m-1}\cdots\alpha_{l,0}}{2^m(\beta_{l,m} - \rho_l(\lambda))\,\beta_{l,m}\cdots(\beta_{l,1} - \rho_l(\lambda))\,\beta_{l,1}}, \qquad \pm m \geqslant 1$$

which vanishes quickly as $|m|$ increases. Therefore,

$$\text{span}\{\cos(lx-ky), \cos lx, \cos(lx+ky)\}, \qquad l=1,\dots,k-1$$

are key subspaces of $H^2_{1,k}$ to study the spectral properties of the operator $-\Delta+\lambda A_k$. Additionally, noting that the fluid motion described by Eq. (19) is excited by the spatial force $-k\cos ky$, we thus choose

$$\text{span}\{\cos ky, \cos(lx-ky), \cos lx, \cos(lx+ky)\}, \qquad l=1,\dots,k-1 \quad (20)$$

as the desired the subspaces of $H^2_{1,k}$. Equation (19) in $H^2_{1,k}$ is projected onto these spaces as follows:

$$\phi_1=\cos ky, \qquad \phi_2=\cos(lx-ky), \qquad \phi_3=\cos lx, \qquad \phi_4=\cos(lx+ky)$$
$$\psi=X_1(t)\,\phi_1+X_2(t)\,\phi_2+X_3(t)\,\phi_3+X_4(t)\,\phi_4$$

for every $l=1,\dots,k-1$; the truncation of all the terms of Eq. (19) orthogonal to the space defined in Eq. (20) gives the following result:

$$(\partial_t\psi-\Delta\psi+\lambda\Delta^{-1}(\partial_y\psi\,\partial_x\Delta\psi-\partial_x\psi\,\partial_y\Delta\psi), \phi_n)=(-k\cos ky, \phi_n)$$

for $n=1, 2, 3, 4$, where (ϕ, φ) denotes the inner product $\int_{T^2}\phi\varphi\,dx\,dy$. After algebraic manipulation, this produces the set of coupled equations

$$\frac{dX_1}{dt}+k^2X_1+\frac{\lambda lk}{2}X_3(X_2-X_4)=-k$$

$$\frac{dX_2}{dt}+(l^2+k^2)\,X_2-\frac{\lambda kl(k^2-l^2)}{2(l^2+k^2)}X_1X_3=0$$

$$\frac{dX_3}{dt}+l^2X_3-\frac{\lambda lk}{2}X_1(X_2-X_4)=0$$

$$\frac{dX_4}{dt}+(l^2+k^2)\,X_4+\frac{\lambda kl(k^2-l^2)}{2(l^2+k^2)}X_1X_3=0$$

from which we see that

$$\frac{d(X_2+X_4)}{dt}+(l^2+k^2)(X_2+X_4)=0$$

This shows that $X_4(t)+X_2(t)$ decay exponentially, and by letting $X_4=-X_2$, we obtain the following coupled equations describing a three-dimensional dynamical system:

$$\frac{dX_1}{dt}+k^2X_1+\lambda lkX_3X_2=-k$$

$$\frac{dX_2}{dt}+(l^2+k^2)\,X_2-\frac{\lambda kl(k^2-l^2)}{2(l^2+k^2)}\,X_1X_3=0 \tag{21}$$

$$\frac{dX_3}{dt}+l^2X_3-\lambda lkX_1X_2=0$$

for $l=1,\ldots,k-1$. As deduced in Section 2, we denote by φ^t the dynamical system such that $\varphi^t(\mathbf{X}(0))=\mathbf{X}(t)=(X_1(t), X_2(t), X_3(t))$, and denote by $D\varphi^t(\mathbf{X}(0))$ the Fréchet derivative of the operator φ^t with respect to the initial data $\mathbf{X}(0)$. It follows from Eq. (21) that

$$(d/dt)\,D\varphi^t(\mathbf{X}(0))\,h^\perp=M(\varphi^t(\mathbf{X}(0)))\,D\varphi^t(\mathbf{X}(0))\,h^\perp$$

$$D\varphi^0(\mathbf{X}(0))\,h^\perp=h^\perp,\qquad h=(h_1,h_2,h_3)\in R^3$$

where

$$M(\mathbf{X})=\begin{pmatrix} -k^2 & -\lambda lkX_3 & -\lambda lkX_2 \\ \dfrac{\lambda kl(k^2-l^2)}{2(l^2+k^2)}X_3 & -(l^2+k^2) & \dfrac{\lambda kl(k^2-l^2)}{2(l^2+k^2)}X_1 \\ \lambda klX_2 & \lambda klX_1 & -l^2 \end{pmatrix}$$

In particular, the matrix

$$M((-1/k,0,0))=\begin{pmatrix} -k^2 & 0 & 0 \\ 0 & -l^2-k^2 & -\dfrac{\lambda l(k^2-l^2)}{2(l^2+k^2)} \\ 0 & -\lambda l & -l^2 \end{pmatrix}$$

has eigenvalues

$$\sigma_1^*=-k^2$$

$$\sigma_2^*=-\frac{2l^2+k^2+[k^4+2\lambda^2l^2(k^2-l^2)/(l^2+k^2)]^{1/2}}{2}$$

$$\rho_l^*(\lambda)=\frac{-2l^2-k^2+[k^4+2\lambda^2l^2(k^2-l^2)/(l^2+k^2)]^{1/2}}{2}$$

On setting $\lambda_l^* = [2(k^2+l^2)^2/(k^2-l^2)]^{1/2}$, we find that $\sigma_1^*, \sigma_2^* < 0$ for all $\lambda > 0$ and

$$\rho_l^*(\lambda) = \begin{cases} <0 & \text{when} \quad \lambda < \lambda_l^* \\ =0 & \text{when} \quad \lambda = \lambda_l^* \\ >0 & \text{when} \quad \lambda > \lambda_l^* \end{cases}$$

It is easy to verify that Eq. (21) with $\lambda > \lambda_l^*$ has exactly two other equilibrium points

$$\mathbf{Y} = (Y_1, Y_2, Y_3) = \left(-\frac{\lambda_l^*}{k\lambda}, \frac{1}{\lambda}\left(\frac{\lambda}{\lambda_l^*} - 1\right)^{1/2}, -\frac{\lambda_l^*}{l\lambda}\left(\frac{\lambda}{\lambda_l^*} - 1\right)^{1/2} \right)$$

$$\mathbf{Z} = (Z_1, Z_2, Z_3) = \left(-\frac{\lambda_l^*}{k\lambda}, -\frac{1}{\lambda}\left(\frac{\lambda}{\lambda_l^*} - 1\right)^{1/2}, \frac{\lambda_l^*}{l\lambda}\left(\frac{\lambda}{\lambda_l^*} - 1\right)^{1/2} \right)$$

By denoting $p(\rho) = \det(\rho I - M(\mathbf{Z}))$ the characteristic polynomial of the matrix M at the nontrivial equilibrium point $\mathbf{Z}$, we obtain, for $\lambda > \lambda_l^*$,

$$\begin{aligned}
p(\rho) &= \det(\rho I - M(\mathbf{Y})) \\
&= (k^2+\rho)\left((l^2+k^2+\rho)(l^2+\rho) - \frac{(\lambda kl)^2 (k^2-l^2)}{2(k^2+l^2)} Y_1^2 \right) \\
&\quad + \lambda kl Y_3 \left(\frac{\lambda kl(k^2-l^2)(l^2+\rho)}{2(k^2+l^2)} Y_3 + \frac{(\lambda kl)^2 (k^2-l^2)}{2(k^2+l^2)} Y_1 Y_2 \right) \\
&\quad + \lambda kl Y_2 \left(\frac{(\lambda kl)^2 (k^2-l^2)}{2(k^2+l^2)} Y_1 Y_3 + (l^2+k^2+\rho)\, \lambda kl Y_2 \right) \\
&= (k^2+\rho)(l^2+k^2+\rho)(l^2+\rho) - \left(\frac{\lambda kl}{\lambda_l^*}\right)^2 (k^2+l^2)(k^2+\rho)\, Y_1^2 \\
&\quad + \left(\frac{\lambda kl}{\lambda_l^*}\right)^2 (k^2+l^2)(l^2+\rho)\, Y_3^2 + 2\left(\frac{\lambda kl}{\lambda_l^*}\right)^2 (k^2+l^2)\, \lambda kl Y_1 Y_2 Y_3 \\
&\quad + (\lambda kl)^2 (k^2+l^2+\rho)\, Y_2^2 \\
&= (k^2+\rho)(l^2+k^2+\rho)(l^2+\rho) + l^2(k^2+l^2)(k^2+\rho) \\
&\quad + k^2(k^2+l^2)(l^2+\rho)\left(\frac{\lambda}{\lambda_l^*} - 1\right) + 2k^2l^2(k^2+l^2)\left(\frac{\lambda}{\lambda_l^*} - 1\right)
\end{aligned}$$

$$+k^2l^2(k^2+l^2+\rho)\left(\frac{\lambda}{\lambda_l^*}-1\right)$$

$$=\rho^3+2(k^2+l^2)\,\rho^2+\frac{\lambda}{\lambda_l^*}k^2(k^2+2l^2)\,\rho+4k^2l^2(k^2+l^2)\left(\frac{\lambda}{\lambda_l^*}-1\right)$$

Now **Y** and **Z** can only become unstable equilibrium points when $p(\rho)$ has a root on the imaginary axis $\{\rho \in C \mid \operatorname{Re}\rho = 0\}$, which, however, contradicts the observation

$$2(k^2+l^2)\frac{\lambda}{\lambda_l^*}k^2(k^2+2l^2)>4k^2l^2(k^2+l^2)\left(\frac{\lambda}{\lambda_l^*}-1\right)>0, \qquad \lambda>\lambda_l^*$$

Thus, we find that the eigenvalues of the matrices $M(\mathbf{Y})$ and $M(\mathbf{Z})$ remain in the half complex plane $\{\rho \in C \mid \operatorname{Re}\rho < 0\}$ for all $\lambda > \lambda_l^*$. This derivation allows the following summary:

Theorem 3.1. Equation (21) with $l = 1,..., k-1$ possesses a unique bifurcation point

$$(\lambda_l^*, (-1/k, 0, 0)), \qquad \lambda_l^* = [2(l^2+k^2)^2/(k^2-l^2)]^{1/2}$$

and this is a pitchfork bifurcation point. The global attractor of Eq. (21) is the single point $\{(-1/k, 0, 0)\}$ when $\lambda \leqslant \lambda_l^*$, and consists of two heteroclinic orbits joined at the saddle equilibrium point $(-1/k, 0, 0)$ when $\lambda > \lambda_l^*$.

Based on this theorem, now we give phase portraits for the global attractors of Eq. (21) for $k = 2, 3$ as examples.

Rewriting Eq. (21) in the form $d\mathbf{X}/dt = f_{l,k,\lambda}(\mathbf{X})$, $\mathbf{X} = (X_1, X_2, X_3)$, we can discretize Eq. (21) by the four-step Adams–Bashforth method (see, for example, ref. 11) to obtain

$$\mathbf{X}_{n+1} = \mathbf{X}_n + \frac{h}{24}(55 f_{l,k,\lambda}(\mathbf{X}_n) - 59 f_{l,k,\lambda}(\mathbf{X}_{n-1})$$
$$+ 37 f_{l,k,\lambda}(\mathbf{X}_{n-2}) - 9 f_{l,k,\lambda}(\mathbf{X}_{n-3}))$$

with step length $h = 0.0005$, and $(l, k) = (1, 2)$, $(1, 3)$, and $(2, 3)$. To illustrate the typical solutions to these equations, we shall, for demonstration, take a Reynolds number $\lambda = 50$ or $\lambda = 200$, and take the two initial data

$$\mathbf{X}^{\pm} = (-1/k, 0, \pm 0.001)$$

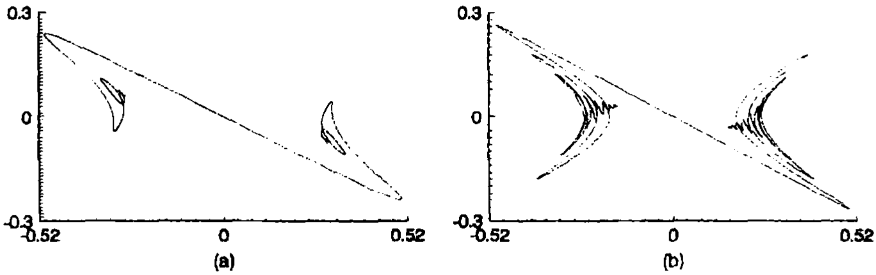

Fig. 1. Phase portraits on the (X_3, X_2) plane for the global attractor or the one-dimensional unstable manifold of the saddle point $(-1/2, 0, 0)$ for Eq. (21) with $k = 2$ and $l = 1$. Here (a) $\lambda = 50$, (b) $\lambda = 200$.

The numerical experiment in the form of phase portraits on the (X_3, X_2) plane is displayed in Fig. 1 for $k = 2$ and in Fig. 2 for $k = 3$. $\mathbf{X}^+$ and $\mathbf{X}^-$ are very close to the unstable manifold of the equilibrium point $(-1/k, 0, 0)$, and so the two orbits produced by $\mathbf{X}^+$ and $\mathbf{X}^-$ can be regarded as the two heteroclinic orbits connected at the point $(-1/k, 0, 0)$. From Theorem 3.1 it follows that the unstable manifold of $(-1/k, 0, 0)$ consisting of these two heteroclinic orbits joined at the saddle

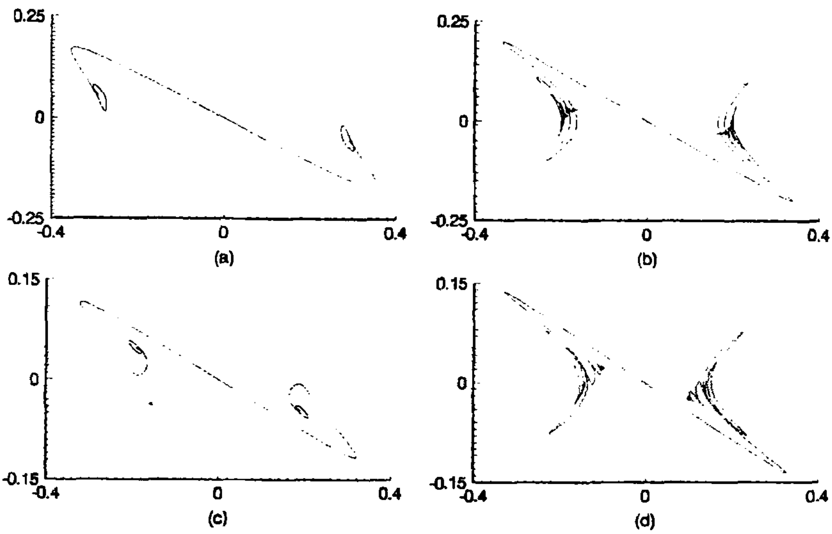

Fig. 2. Phase portraits on the (X_3, X_2) plane for the global attractor or the one-dimensional unstable manifold of the saddle point $(-1/2, 0, 0)$ of Eq. (21) with $k = 3$. Here (a) $(l, \lambda) = (1, 50)$, (b) $(l, \lambda) = (1, 200)$, (c) $(l, \lambda) = (2, 50)$, (d) $(l, \lambda) = (2, 200)$.

point $\mathbf{X}_0$ is the global attractor of Eq. (21). From Theorem 3.1 or from the above numerical experiment it is readily seen that the topological structure of the attractor remains unchanged as λ increases.

4. 17-MODE TRUNCATION SCHEME

To support the stable results deduced in the previous section, we shall provide a 17-mode truncation model for Eq. (19) in $H^2_{l,k}$ and provide numerical experiments for this model for $k=2$, 3 showing corroborative evidence to the findings of Section 3. Before computation, let us analyze this new system to strengthen the credence of the numerical results. It is useful to note that every solution of Eq. (19) in $H^2_{l,k}$ is of the form

$$\psi = \sum_{n=1}^{\infty} \xi_n(t) \cos kny + \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \eta_{m,n}(t) \cos(mlx + kny)$$

This Navier–Stokes flow is excited by the spatial force $-k \cos ky$, and the pitchfork bifurcation stems from the term $\sum_{n=-\infty}^{\infty} \eta_{1,n} \cos(lx + kny)$, which then influences the terms $\sum_{n=-\infty}^{\infty} \eta_{2,n} \cos(2lx + kny)$ and $\sum_{n=1}^{\infty} \xi_n \cos kny$. Furthermore, the modes $\cos(lx + kny)$ and $\cos(2lx + kny)$ have limited symmetry with the modes $\cos(lx - kny)$ and $\cos(2lx - kny)$, respectively, and by numerical experiments it can be shown that ξ_n, $\eta_{1,n}$, and $\eta_{1,-n}$ are of order 10^{-n-1}, and that $\eta_{2,n}$ and $\eta_{2,-n}$ are of order 10^{-n-2} for $n \geqslant 0$. For these reasons, this Navier–Stokes flow can be adequately approximated by a function in the form

$$\sum_{n=1}^{N} \xi_n(t) \cos kny + \sum_{n=-N}^{N} \xi_{1,n}(t) \cos(lx + kny)$$
$$+ \sum_{n=-N}^{N} \zeta_{2,n}(t) \cos(2lx + kny) \tag{22}$$

for $N > 1$. Here, for simplification, we take $N = 3$. By the previous reasoning, Eq. (19) in $H^2_{l,k}$ is projected onto the 17-dimensional space

$$\Pi_{l,k} = \operatorname{span}\{\cos kny, \cos(lx + mky), \cos(2lx + mky) \mid n = 1, 2, 3,$$
$$m = -3, \ldots, 3\}$$

Navier–Stokes flow is truncated to the following form:

$$\psi(t) = \sum_{j=1}^{3} X_j(t)\cos kjy + \sum_{n=4}^{10} X_n(t)\cos(lx + k(n-7)\,y) + \sum_{m=11}^{17} X_m(t)\cos(2lx + k(m-14)\,y)$$

On setting, for $1 \leqslant j \leqslant 3$, $4 \leqslant n \leqslant 10$, and $11 \leqslant m \leqslant 17$,

$$a_j = k^2 j^2, \qquad a_n = l^2 + k^2(n-7)^2, \qquad a_m = 4l^2 + k^2(m-14)^2$$

$$\phi_j = \sin kjy, \qquad \phi_n = \sin(lx + k(n-7)\,y), \qquad \phi_m = \sin(2lx + k(m-14)\,y)$$

$$\psi_j = \cos kjy, \qquad \psi_n = \cos(lx + k(n-7)\,y), \qquad \psi_m = \cos(2lx + k(m-14)\,y)$$

we have

$$-\partial_y \psi = \sum_{j=1}^{3} kjX_j(t)\,\phi_j + \sum_{n=4}^{10} k(n-7)\,X_n(t)\,\phi_n + \sum_{m=11}^{17} k(m-14)\,X_m(t)\,\phi_m$$

$$\partial_x \Delta\psi = \sum_{n=4}^{10} la_n X_n(t)\,\phi_n + \sum_{m=11}^{17} 2la_m X_m(t)\,\phi_m$$

$$-\partial_x \psi = \sum_{n=4}^{10} lX_n(t)\,\phi_n + \sum_{m=11}^{17} 2lX_m(t)\,\phi_m$$

$$\partial_y \Delta\psi = \sum_{j=1}^{3} kja_j X_j(t)\,\phi_j + \sum_{n=4}^{10} k(n-7)\,a_n X_n(t)\,\phi_n + \sum_{m=11}^{17} k(m-14)\,a_m X_m(t)\,\phi_m$$

and so, after the necessary manipulations, the nonlinear contribution is given by

$$\partial_y \psi\, \partial_x \dot{\Delta}\psi - \partial_x \psi\, \partial_y \Delta\psi = \sum_{j=1}^{3} \frac{klj}{2}\left(\sum_{n=4+j}^{10} (a_n - a_{n-j})\,X_n X_{n-j} + 2\sum_{m=11+j}^{17} (a_m - a_{m-j})\,X_m X_{m-j}\right)\psi_j$$

$$+\sum_{j=1}^{3}\frac{klj}{2}\left(\sum_{n=4}^{10-j}(a_j-a_{n+j})\,X_jX_{n+j}\psi_n\right.$$

$$\left.-\sum_{n=4+j}^{10}(a_j-a_{n-j})\,X_jX_{n-j}\psi_n\right)$$

$$+\sum_{m=11}^{17}\sum_{4\leqslant n,\,m-n\leqslant 10}\frac{kl(2n-m)}{2}(a_m-a_{m-n})\,X_mX_{m-n}\psi_n$$

$$+\sum_{j=1}^{3}klj\left(\sum_{m=11}^{17-j}(a_j-a_{m+j})\,X_jX_{m+j}\psi_m\right.$$

$$\left.-\sum_{m=11+j}^{17}(a_j-a_{m-j})\,X_jX_{m-j}\psi_m\right)$$

$$-\sum_{m=11}^{17}\sum_{4\leqslant n,\,m-n\leqslant 10}\frac{kl(2n-m)}{2}a_nX_{m-n}X_n\psi_m+E$$

where E is a term orthogonal to the 17-dimensional space. Equation (19) in $H^2_{l,k}$ is thus truncated to the following coupled set of 17 ordinary differential equations:

$$0=\sum_{i=1}^{17}\int_{T^2}\left(\frac{d\psi}{dt}-\Delta\psi+B_{k,\lambda}(\psi)\right)\psi_i\,dx\,dy\,\psi_i$$

which on expansion gives

$$0=\sum_{i=1}^{17}\frac{dX_i}{dt}\psi_i+\sum_{i=1}^{17}a_iX_i\psi_i+k\psi_1$$

$$=\sum_{j=1}^{3}\frac{klj}{2}\left(\sum_{n=4+j}^{10}\frac{a_n-a_{n-j}}{a_j}X_nX_{n-j}+2\sum_{m=11+j}^{17}\frac{a_m-a_{m-j}}{a_j}X_mX_{m-j}\right)\psi_j$$

$$+\sum_{j=1}^{3}\frac{klj}{2}\left(\sum_{n=4}^{10-j}\frac{a_j-a_{n+j}}{a_n}X_jX_{n+j}\psi_n-\sum_{n=4+j}^{10}\frac{a_j-a_{n-j}}{a_n}X_jX_{n-j}\psi_n\right)$$

$$+\sum_{m=11}^{17}\sum_{4\leqslant n,\,m-n\leqslant 10}\frac{kl(2n-m)(a_m-a_{m-n})}{2a_n}X_mX_{m-n}\psi_n$$

$$+\sum_{j=1}^{3}klj\left(\sum_{m=11}^{17-j}\frac{a_j-a_{m+j}}{a_m}X_jX_{m+j}\psi_m\right.$$

$$- \sum_{m=11+j}^{17} \frac{a_j - a_{m-j}}{a_m} X_j X_{m-j} \psi_m \Bigg)$$

$$- \sum_{m=11}^{17} \sum_{4 \leqslant n,\, m-n \leqslant 10} \frac{kl(2n-m)\, a_n}{2a_m} X_{m-n} X_n \psi_m$$

The rewriting of this coupled set of equations in the form

$$\frac{d\mathbf{X}}{dt} = F_{l,k,\lambda}(\mathbf{X}), \qquad \mathbf{X} = (X_1, ..., X_{17}) \tag{23}$$

leads itself to a discretization by the four-step Adams–Bashforth method to give

$$\mathbf{X}_{n+1} = \mathbf{X}_n + \frac{h}{24}(55F_{l,k,\lambda}(\mathbf{X}_n) - 59F_{l,k,\lambda}(\mathbf{X}_{n-1})$$
$$+ 37F_{l,k,\lambda}(\mathbf{X}_{n-2}) - 9F_{l,k,\lambda}(\mathbf{X}_{n-3}))$$

where a step length $h = 0.0002$ is again chosen.

Similar to Section 3, now we take $(l, k) = (1, 2)$, $(1, 3)$, and $(2, 3)$ as examples to display numerical experiments on global attractors of the truncated model in Fig. 3 for $k = 2$ and in Fig. 4 for $k = 3$ through their phase portraits on the (X_7, X_6) plane for the Reynolds number $\lambda = 50$ or $\lambda = 200$. As in Section 3, the discretization starts respectively from the two initial data

$$\mathbf{X}^{\pm} = (-1/k, X_2, ..., X_6, \pm 0.001, X_8, ..., X_{17}), \qquad X_n = 0$$

$\mathbf{X}^+$ and $\mathbf{X}^-$ are very close to the saddle point $\mathbf{X}_0 = (-1/k, X_2, ..., X_{17})$ with $X_n = 0$, the projection of $-(1/k)\cos ky$ on $\Pi_{l,k}$, but are not in the stable

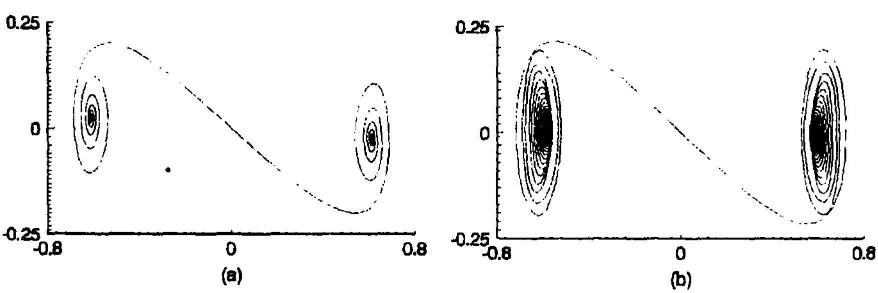

Fig. 3. Phase portraits on the (X_7, X_6) plane for the global attractor or the one-dimensional unstable manifold of $\mathbf{X}_0$ for the 17-mode truncation model with $l = 1$ and $k = 2$. The two stable equilibrium points are $\mathbf{X}^+_{1,2}$ and $\mathbf{X}^-_{1,2}$. Here (a) $\lambda = 50$, (b) $\lambda = 200$.

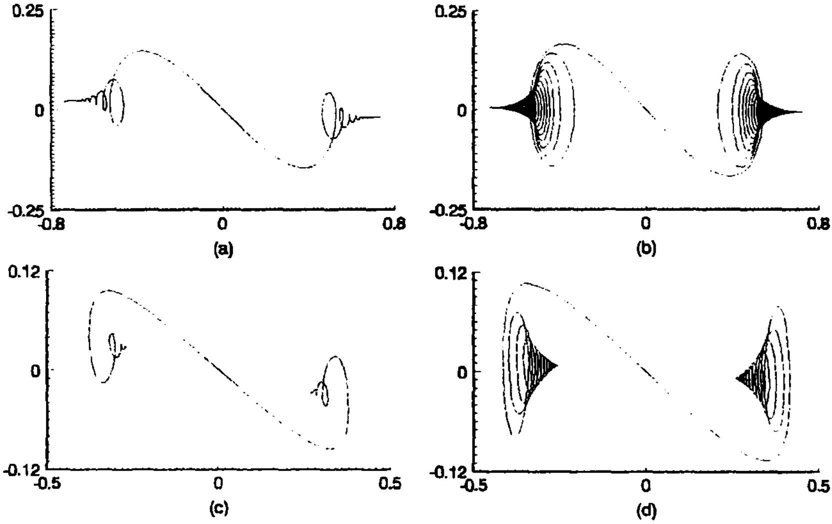

Fig. 4. Phase portraits on the (X_7, X_6) plane for the two typical heteroclinic orbits joined at the saddle point $\mathbf{X}_0$ and ending respectively at the stable equilibrium points $\mathbf{X}^+_{l,3}$ and $\mathbf{X}^-_{l,3}$. Here (a) $(l, k, \lambda) = (1, 3, 50)$, (b) $(l, k, \lambda) = (1, 3, 200)$, (c) $(l, k, \lambda) = (2, 3, 50)$, (d) $(l, k, \lambda) = (2, 3, 200)$.

manifold of $\mathbf{X}_0$. We find that the discrete flow from $\mathbf{X}^+$ (resp. $\mathbf{X}^-$) approaches a stable equilibrium point in $\Pi_{l,k}$, which denoted by $\mathbf{X}^+_{l,k}$ (resp. $\mathbf{X}^-_{l,k}$).

It should be noted that many other discrete orbits with initial data close to the attractor are examined. They approaches either $\mathbf{X}^+_{l,k}$ or $\mathbf{X}^-_{l,k}$.

From Figs. 3 and 4 together with Theorems 2.1 and 3.1 we readily see that $\mathbf{X}^\pm_{1,2}$ (resp. $\mathbf{X}^\pm_{2,3}$) are the only equilibrium points in $\Pi_{1,2}$ (resp. $\Pi_{2,3}$) bifurcated from $\mathbf{X}_0$. Moreover, there are two pitchfork bifurcation values $\lambda^*_1 < \lambda^*_2$ for Eq. (23) with $(l, k) = (1, 3)$. The global attractor of the 17-mode truncation model in $\Pi_{1,3}$ coincides with $\mathbf{X}_0$ when $0 < \lambda < \lambda^*_1$, contains the two stable equilibrium points $\mathbf{X}^\pm_{1,3}$ when $\lambda^*_1 < \lambda < \lambda^*_2$, and conains the two stable equilibrium points $\mathbf{X}^\pm_{1,3}$ and the two saddle points $\mathbf{X}^\pm_{2,3}$ having respectively a one-dimensional unstable manifold in $\Pi_{1,3}$ when $\lambda > \lambda^*_2$. From Figs. 3 and 4 we see that the orbits generated by the initial value $\mathbf{X}^+$ (resp. $\mathbf{X}^-$) are almost the heteroclinic orbits starting from $\mathbf{X}_0$ and ending at $\mathbf{X}^+_{l,k}$ (resp. $\mathbf{X}^-_{l,k}$), and the stability of $\mathbf{X}^\pm_{l,k}$ in $\Pi_{l,k}$ remains unchanged as λ increases.

Thus, on the condition that the 17-mode truncation model is a suitable approximation of Eq. (19) in $H^2_{l,k}$, we find that the topological structure of global attractors of Eq. (19) in $H^2_{l,k}$ and Eq. (21) with

$(l, k) = (1, 2)$, $(2, 3)$ is the same. Since $H^2_{2,3} \subset H^2_{1,3}$, from Theorem 2.1 we see that there are two pitchfork bifurcation values $0 < \lambda_1 < \lambda_2$. When $0 < \lambda < \lambda_1$, the global attractor of Eq. (19) in $H^2_{1,3}$ coincides with the steady-state solution $\psi_0 = -(1/3)\cos 3y$. When $\lambda_1 < \lambda < \lambda_2$, ψ_0 has a one-dimensional unstable manifold and the two bifurcated steady-state solutions are stable in $H^2_{1,3}$. When $\lambda > \lambda_2$, ψ_0 has a two-dimensional unstable manifold in $H^2_{1,3}$. Each of the second pair of bifurcated steady-state solutions is stable in $H^2_{2,3}$ and has a one-dimensional unstable manifold in $H^2_{1,3}$.

This additionally confirms the trends observed in the four-mode truncation model in Section 3.

5. HOPF BIFURCATION

In previous sections, we gave an analysis and numerical experiments on pitchfork bifurcation and stability for bifurcated solutions of Eq. (19) in the special invariant subspaces $H^2_{l,k}$. In order to provide more evidence for the fluid motion outside these invariant spaces, we shall take $k = 2$ as an example and study the long-time behavior of the Navier–Stokes flow in the subspace $\mathscr{H}^2_{l,2}$ instead of $H^2_{l,2}$. $\mathscr{H}^2_{l,2}$ is also a flow-invariant subspace.

Lemma 5.1. For every $\lambda > 0$ and every initial state $\psi_0 \in \mathscr{H}^2_{l,2}$, Eqs. (5) and (6) admit a unique global solution $\psi \in C([0, \infty); \mathscr{H}^2_{l,2})$.

This lemma is deduced in completely the same way as Lemma 2.1. Thus we reduce Eqs. (5)–(6) in $\mathscr{H}^2_{1,2}$:

$$\frac{d\psi}{dt} - \Delta\psi + B_{2,\lambda}(\psi) = 0, \qquad \psi = \psi(t, \cdot) \in \mathscr{H}^2_{l,2} \tag{24}$$

We shall find that the steady-state solution $-(1/2)\cos 2y$ will lose stability, Hopf bifurcation arises for this equation, and the bifurcated time-dependent periodic solution is likely stable in $\mathscr{H}^2_{1,2}$ when λ varies.

We recall the operator $-\Delta + \lambda A_2 = -\Delta + \lambda\Delta^{-1}\sin 2y\,(\Delta + 4)\,\partial_x$, the Fréchet derivative of the operator $-\Delta + \lambda B_{2,\lambda}$ at $(-1/2)\cos 2y$. As is well known (see, for example, refs. 7 and 15), Hopf bifurcation is essentially based on the existence of a simple pair of conjugate eigenvalues of $-\Delta + \lambda A_2$ crossing the imaginary line as λ increases. This reads as follows:

Lemma 5.2. For $\lambda > 0$, the operator $-\Delta + \lambda A_2$ in $\mathscr{H}^2_{1,2}$ has a eigenvalue $\rho(\lambda)$ with $\operatorname{Im}\rho(\lambda) \neq 0$ such that for some constants $\lambda_1 > \delta > 0$,

$$\operatorname{Re}\rho(\lambda) < 0 \qquad \text{when} \quad 0 < \lambda < \delta$$

$$\operatorname{Re}\rho(\lambda_1) = 0, \quad \operatorname{Re}\rho(\lambda) > 0 \qquad \text{when} \quad \lambda_1 < \lambda < \lambda_1 + \delta$$

and furthermore, the operator $-\Delta + \lambda A_2$ in $\mathscr{H}^2_{l,2}$ with $l \neq 1$ has no eigenvalue ρ satisfying $\operatorname{Re} \rho \leqslant 0$.

Here, without loss of generality, we have supposed that $\mathscr{H}^2_{l,2}$ are complex spaces.

For the special case $k=2$, a similar result in the whole space was obtained by Liu (see, for example, ref. 13). However, we shall provide an alternative and simpler approach for this lemma by following ref. 3, where we examined time-dependent periodic solutions to Eqs. (1)–(2) with k an even number.

Proof. First, from Corollary 2.2 we see that the spectral problem

$$-\Delta\psi + \lambda A_2 \psi = \rho\psi \qquad \text{in} \quad \mathscr{H}^2_{l,2} \quad (\operatorname{Re} \rho \leqslant 4) \tag{25}$$

has no eigenfunction in the form $\psi = \sum_{n=-\infty}^{\infty} \zeta_n \cos(2mlx + 2ny)$ for $l \geqslant 0$, $m \geqslant 0$ and $\operatorname{Re} \rho \leqslant 0$. As in the proof of Lemma 2.2, an eigenfunction ψ to this spectral problem can be supposed in the following form:

$$\psi = \sum_{n=-\infty}^{\infty} \zeta_n \cos(lx + 2ny + y), \qquad l \geqslant 0$$

Thus Eq. (25) with this eigenfunction becomes

$$\frac{\lambda \hat{\alpha}_{l,m-1} \xi_{l,m-1}}{2\hat{\beta}_{l,m}} - \frac{\lambda \hat{\alpha}_{l,m+1} \xi_{l,m+1}}{2\hat{\beta}_{l,m}} + (\hat{\beta}_{l,m} - \rho)\,\xi_{l,m} = 0 \tag{26}$$

for any integer m, where

$$\hat{\beta}_{l,m} = l^2 + (2m+1)^2 \qquad \text{and} \qquad \hat{\alpha}_{l,m} = l(l^2 + (2m+1)^2 - 4)$$

The assertion in the case $l=0$ follows immediately from Eq. (26). Proceeding to the remaining case $l \geqslant 1$, we also set

$$\hat{\gamma}_{l,m} = \frac{\hat{\alpha}_{l,m}\xi_{l,m}}{\hat{\alpha}_{l,m-1}\xi_{l,m-1}}, \quad m \geqslant 0; \qquad \hat{\gamma}_{l,-m} = \frac{\hat{\alpha}_{l,-m}\xi_{l,-m}}{\hat{\alpha}_{l,1-m}\xi_{l,1-m}}, \quad m \geqslant 1 \tag{27}$$

Thus we see that Eq. (26) is equivalent to the following system of algebraic equations:

$$\frac{2(\hat{\beta}_{l,m} - \rho)\,\hat{\beta}_{l,m}}{\lambda \hat{\alpha}_{l,m}} + \frac{1}{\hat{\gamma}_{l,m}} = \hat{\gamma}_{l,m+1} \qquad \text{for} \quad m \geqslant 0$$

$$\frac{2(\hat{\beta}_{l,-m} - \rho)\,\hat{\beta}_{l,-m}}{\lambda \hat{\alpha}_{l,-m}} + \hat{\gamma}_{l,-m-1} = \frac{1}{\hat{\gamma}_{l,-m}} \qquad \text{for} \quad m \geqslant 1$$

This together with the boundedness of $\hat{\gamma}_{l,\pm m}$ from the fact $\psi \in H^2$ implies

$$\hat{\gamma}_{l,m} = \cfrac{-1}{\cfrac{2(\hat{\beta}_{l,m}-\rho)\,\hat{\beta}_{l,m}}{\lambda\hat{\alpha}_{l,m}} + \cfrac{1}{\cfrac{2(\hat{\beta}_{l,m+1}-\rho)\,\hat{\beta}_{l,m+1}}{\lambda\hat{\alpha}_{l,m+1}} + \cfrac{1}{\ddots}}}, \qquad \text{for} \quad m \geqslant 0$$

$$\hat{\gamma}_{l,m} = \cfrac{1}{\cfrac{2(\hat{\beta}_{l,m}-\rho)\,\hat{\beta}_{l,m}}{\lambda\hat{\alpha}_{l,m}} + \cfrac{1}{\cfrac{2(\hat{\beta}_{l,m-1}-\rho)\,\hat{\beta}_{l,m-1}}{\lambda\hat{\alpha}_{l,m-1}} + \cfrac{1}{\ddots}}}, \qquad \text{for} \quad m \leqslant -1$$

Thus if (ψ, ρ) is a solution to the spectral problem (25), the eigenfunctions ψ form a one-dimensional space, since Eq. (27) gives for an arbitrary constant c

$$\xi_{l,0} = c, \qquad \xi_{l,\pm m} = c\,\frac{\hat{\gamma}_{l,\pm m} \cdots \hat{\gamma}_{l,\pm 1}\hat{\alpha}_{l,0}}{\hat{\alpha}_{l,\pm m}}, \qquad m \geqslant 1$$

On the other hand, $\hat{\beta}_{l,m} = \hat{\beta}_{l,-1-m}$ and $\hat{\alpha}_{l,m} = \hat{\alpha}_{l,-1-m}$ for $m \geqslant 0$ yield $\hat{\gamma}_{l,0} = -\hat{\gamma}_{l,-1}$, and Eq. (27) gives $\hat{\gamma}_{l,0} = 1/\hat{\gamma}_{l,-1}$. Hence $\hat{\gamma}_{l,0} = i$, and so Eq. (25) becomes

$$\cfrac{-1}{\cfrac{2\hat{\beta}_{l,0}(\hat{\beta}_{l,0}-\rho)}{\lambda\hat{\alpha}_{l,0}} + \cfrac{1}{\cfrac{2(\hat{\beta}_{l,1}-\rho)\,\hat{\beta}_{l,1}}{\lambda\hat{\alpha}_{l,1}} + \cfrac{1}{\cfrac{2(\hat{\beta}_{l,2}-\rho)\,\hat{\beta}_{l,2}}{\lambda\hat{\alpha}_{l,2}} + \cfrac{1}{\ddots}}}} = i$$

If $l \geqslant 2$ and $\operatorname{Re}\rho \leqslant 4$, the real part of the left-hand side of this equation is negative. Therefore we need only consider the case $l = 1$, that is,

$$-\frac{2(2-\rho)}{\lambda} + \cfrac{1}{\cfrac{2\beta_1(\beta_1-\rho)}{\alpha_1\lambda} + \cfrac{1}{\cfrac{2\beta_2(\beta_2-\rho)}{\alpha_2\lambda} + \cfrac{1}{\ddots}}} = i \tag{28}$$

with

$$\beta_n = \hat{\beta}_{1,n} \qquad \text{and} \qquad \alpha_n = \hat{\alpha}_{1,n}$$

Second, to show the existence of the eigenvalue ρ with $\operatorname{Re}\rho \leqslant 2$, we multiply Eq. (28) by λ to obtain

$$-4 - i\lambda + \cfrac{1}{\cfrac{\beta_1(2\beta_1 - 2\rho)}{\alpha_1 \lambda^2} + \cfrac{1}{\cfrac{\beta_2(2\beta_2 - 2\rho)}{\alpha_2} + \cfrac{1}{\ddots}}} = -2\rho \tag{29}$$

Let

$$\mu = -2\operatorname{Re}\rho, \qquad \nu = -2\operatorname{Im}\rho$$

$$\Phi_\lambda(\mu, \nu) = (\operatorname{Re}\Psi_\lambda(\mu, \nu), \operatorname{Im}\Psi_\lambda(\mu, \nu))$$

where $\Psi_\lambda(\mu, \nu)$ denotes the left-hand side of Eq. (29). This equation is solved when a suitable value is found for $(\mu, \nu) = (\mu(\lambda), \nu(\lambda))$ a fixed point of Φ_λ.

From Eq. (29) it follows that $\operatorname{Re}\Psi_\lambda(\mu, \nu) > -4$ and

$$|\Psi_\lambda(\mu, \nu) + 4| \leqslant \lambda + \frac{\alpha_1 \lambda^2}{\beta_1(2\beta_1 + \mu)} \leqslant \lambda + \lambda^2 \tag{30}$$

for all $\mu \geqslant -4$, $\nu \in R$, and $\lambda > 0$. Hence for $K = 4 + \lambda + \lambda^2$,

$$\Phi_\lambda : [-4, \infty) \times R \to [-4, K] \times [-K, K]$$

That is, Φ_λ maps $[-4, K] \times [-K, K]$ into itself. Note that $\Phi_\lambda(\mu, \nu)$ is continuous with respect to (μ, ν, λ). By the Brouwer fixed-point theorem (see, for example, ref. 16), Φ_λ admits a fixed point $(\mu(\lambda), \nu(\lambda)) \in [-4, K] \times [-K, K]$. We can make a suitable choice for the fixed points of Φ_λ to ensure the continuity of $\mu(\lambda)$ and $\nu(\lambda)$.

Finally, it remains to prove that $\mu(\lambda)$ crosses the zero point. Let us suppose, on the contrary, $-4 < \mu(\lambda) \leqslant 0$ for all $\lambda > 0$. By Eq. (30),

$$\Psi_\lambda(\mu(\lambda), \nu(\lambda)) \to -4 < 0 \qquad \text{as} \quad \lambda \to 0$$

In order to prove $\Psi_\lambda(\mu(\hat\lambda), \nu(\hat\lambda)) > 0$ for some $\hat\lambda > 0$, we introduce the following.

Lemma 5.3. Let $(\mu(\lambda), \nu(\lambda))$ be the fixed point of Φ_λ with $-4 < \mu(\lambda) \leqslant 0$. Then $|\nu(\lambda)| \leqslant 2\lambda$ for all $\lambda > 0$.

The proof of this lemma will be shown at the end of this section.

Without loss of generality, by this lemma we can suppose that

$$\frac{\nu(\lambda)}{\lambda} \to \nu_0 \qquad \text{as} \quad \lambda \to \infty \tag{31}$$

for some constant ν_0. Setting real functions $h = h(\lambda)$ and $g = g(\lambda)$ such that for $(\mu, \nu) = (\mu(\lambda), \nu(\lambda))$,

$$h + ig = \frac{\beta_2(2\beta_2 + \mu + i\nu)}{\alpha_2} + \cfrac{\lambda}{\cfrac{\beta_3(2\beta_3 + \mu + i\nu)}{\alpha_3\lambda} + \cfrac{1}{\cfrac{\beta_4(2\beta_4 + \mu + i\nu)}{\alpha_4\lambda} + \cfrac{1}{\ddots}}}$$

we have

$$h(\lambda) \geqslant \frac{\beta_2(2\beta_2 + \mu(\lambda))}{\alpha_2} \geqslant \frac{\beta_2(2\beta_2 - 4)}{\alpha_2} > 52, \qquad \lambda > 0 \tag{32}$$

and, by Eq. (28),

$$-\frac{4 + \mu + i\nu}{\lambda} + \cfrac{1}{\cfrac{5(20 + \mu + i\nu)}{3\lambda} + \cfrac{1}{\cfrac{h + ig}{\lambda}}} = i$$

and thus

$$\frac{h + ig}{\lambda} = \cfrac{1}{-\cfrac{5(20 + \mu + i\nu)}{3\lambda} + \cfrac{1}{\cfrac{4 + \mu + i\nu}{\lambda} + i}}$$

Since

$$\frac{5c}{3} + \frac{1}{c+1} \neq 0 \qquad \text{for all} \quad c \in R$$

we have, by Eq. (31),

$$\lim_{\lambda \to \infty} \frac{h(\lambda)}{\lambda} = 0, \qquad \lim_{\lambda \to \infty} \frac{g(\lambda)}{\lambda} = \left(\frac{5\nu_0}{3} + \frac{1}{\nu_0 + 1}\right)^{-1} \tag{33}$$

On the other hand, it follows from Eq. (29) that

$$\mu+4=\operatorname{Re}\frac{1}{\dfrac{5(20+\mu+iv)}{3\lambda^2}+\dfrac{1}{h+ig}}$$

$$=\frac{1}{\dfrac{5(20+\mu)}{3\lambda^2}+\dfrac{h}{h^2+g^2}+\dfrac{\left(\dfrac{5v(h^2+g^2)}{3\lambda^2}-g\right)^2}{(h^2+g^2)\left(\dfrac{5(20+\mu)(h^2+g^2)}{3\lambda^2}+h\right)}}$$

$$\geqslant\frac{1}{\dfrac{5(20+\mu)}{3\lambda^2}+\dfrac{h}{h^2+g^2}+\left(\dfrac{5v(h^2+g^2)}{3\lambda^2}-g\right)^2\dfrac{1}{h(h^2+g^2)}}$$

$$=\frac{1}{\dfrac{5(20+\mu)}{3\lambda^2}+\left(\dfrac{5v}{3\lambda^2}\right)^2 h+\dfrac{1}{h}\left(\dfrac{5vg}{3\lambda^2}-1\right)^2}$$

$$\geqslant\frac{1}{\dfrac{35}{\lambda^2}+\left(\dfrac{5v}{3\lambda^2}\right)^2 h+\dfrac{1}{52}\left(\dfrac{5vg}{3\lambda^2}-1\right)^2}\qquad\text{by Eq. (32)}$$

which, by making use of Eqs. (31) and (33) and setting $\lambda\to\infty$, tends to

$$52\left(\frac{5v_0}{3}\lim_{\lambda\to\infty}\frac{g(\lambda)}{\lambda}-1\right)^{-2}$$

$$=52\left(\frac{5v_0}{3}\frac{1}{\dfrac{5v_0}{3}+\dfrac{1}{v_0+1}}-1\right)^{-2}=52\left(\frac{5v_0(v_0+1)}{3}+1\right)^2>13$$

This implies that there exists a constant $\hat{\lambda}>0$ such that $\mu(\hat{\lambda})>9$. This leads to a contradiction, and implies the existence of the desired positive constants λ_1 and δ. The proof is complete.

Now we proceed to the proof of Lemma 5.3.

Proof of Lemma 5.3. For $n \geqslant 0$ and $\lambda > 0$, we set

$$d_n(\lambda) = -\cfrac{1}{\cfrac{\beta_n(2\beta_n + \mu(\lambda) + i\nu(\lambda))}{\alpha_n \lambda} + \cfrac{1}{\cfrac{\beta_{n+1}(2\beta_{n+1} + \mu(\lambda) + i\nu(\lambda))}{\alpha_{n+1}\lambda} + \cfrac{1}{\ddots}}}$$

which gives, for $\lambda > 0$,

$$|d_n(\lambda)| \leqslant \frac{\alpha_n \lambda}{\beta_n(2\beta_n + \mu(\lambda))} \leqslant \frac{\lambda}{2\beta_n - 4} \to 0 \qquad \text{as} \quad n \to \infty \tag{34}$$

and

$$\operatorname{Im} d_{n+1}(\lambda) = \frac{\beta_n \nu(\lambda)}{\alpha_n \lambda} + \operatorname{Im} \frac{1}{d_n(\lambda)}$$

and so

$$|\operatorname{Im} d_{n+1}(\lambda)| \geqslant \frac{|\nu(\lambda)|}{\lambda} - \frac{1}{|\operatorname{Im} d_n(\lambda)|}, \qquad n \geqslant 0 \tag{35}$$

On the contrary, we suppose that the assertion of the lemma is not valid. Then there is a positive value λ' such that $|\nu(\lambda')| > 2\lambda'$. By Eq. (28), $d_0(\lambda') \equiv i$. This together with Eq. (35) implies $|d_n(\lambda')| \geqslant 1$ for all $n \geqslant 0$. This contradicts Eq. (34). The proof is complete.

Based on Lemma 5.2, we now follow the study in the preceding section to provide an 18-mode truncation model for Eq. (24) and give computational result for the bifurcated time-dependent periodic solution, which seems stable in $\mathscr{H}^2_{1,2}$ for all λ.

As in the derivation of Eq. (22), we see that for every

$$\psi = \sum_{n=1}^{\infty} \xi_n(t) \cos 2ny + \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \eta_{m,n}(t) \cos(2mx - x + 2ny + y)$$

$$+ \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \zeta_{m,n}(t) \cos(2mx + 2ny)$$

the solution of Eq. (24) can be adequately approximated by a function in the form

$$\sum_{n=1}^{N} \xi_n(t)\cos 2ny + \sum_{n=-N-1}^{N} \eta_{1,n}(t)\cos(x+2ny+y)$$
$$+\sum_{n=-N}^{N} \zeta_{1,n}(t)\cos(2x+2ny)$$

since the proof of Lemma 5.2 shows that $\eta_{m,n}$ have limit symmetry with respect to $\eta_{m,-n-1}$ for $n \geqslant 0$. From numerical experiments we find that ξ_n, $\eta_{1,n}$, and $\eta_{1,-n-1}$ are of order 10^{-n-1} and $\zeta_{1,n}$ is of order 10^{-n-2} for $n \geqslant 0$. Set $N=3$, and set

$$b_j = 4j^2, \qquad b_n = 1+(2(n-8)+1)^2, \qquad b_m = 4+4(m-15)^2$$
$$\psi_j = \cos 2jy, \qquad \psi_n = \cos(x+2ny+y), \qquad \psi_m = \cos(2x+2ny)$$

for $1 \leqslant j \leqslant 3$, $4 \leqslant n \leqslant 11$, and $12 \leqslant m \leqslant 18$. The Navier–Stokes flow is truncated to the form $\psi = \sum_{i=1}^{18} X_i \psi_i$. Following the procedure for producing the 17-mode truncation model, we thus project Eq. (24) onto the 18-dimensional space span$\{\psi_i \mid i=1,..., 18\}$ to obtain the following 18-mode truncation model:

$$\begin{aligned}
0 &= \sum_{i=1}^{18} \frac{dX_i}{dt}\psi_i + \sum_{i=1}^{18} b_i X_i \psi_i + k\psi_1 \\
&= \sum_{j=1}^{3} \frac{klj}{2}\left(\sum_{n=4+j}^{11} \frac{b_n - b_{n-j}}{b_j} X_n X_{n-j} + 2\sum_{m=12+j}^{18} \frac{b_m - b_{m-j}}{b_j} X_m X_{m-j}\right)\psi_j \\
&\quad + \sum_{j=1}^{3} \frac{klj}{2}\left(\sum_{n=4}^{11-j} \frac{b_j - b_{n+j}}{b_n} X_j X_{n+j}\psi_n - \sum_{n=4+j}^{11} \frac{b_j - b_{n-j}}{b_n} X_j X_{n-j}\psi_n\right) \\
&\quad + \sum_{m=12}^{18} \sum_{4 \leqslant n,\, m-n \leqslant 11} \frac{kl(2n-m)(b_m - b_{m-n})}{2b_n} X_m X_{m-n}\psi_n \\
&\quad + \sum_{j=1}^{3} klj\left(\sum_{m=12}^{18-j} \frac{b_j - b_{m+j}}{b_m} X_j X_{m+j}\psi_m - \sum_{m=12+j}^{18} \frac{b_j - b_{m-j}}{b_m} X_j X_{m-j}\psi_m\right) \\
&\quad - \sum_{m=12}^{18} \sum_{4 \leqslant n,\, m-n \leqslant 11} \frac{kl(2n-m)\, b_n}{2b_m} X_{m-n} X_n \psi_m
\end{aligned}$$

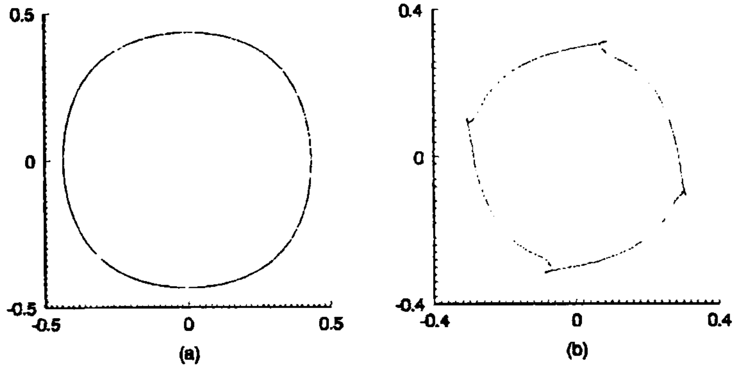

Fig. 5. The (X_8, X_7) phase portraits of the limit cycle solution derived from the 18-mode truncation model for (a) Reynolds number $\lambda = 50$ and (b) Reynolds number $\lambda = 200$.

for $k = 2$ and $l = 1$, which is rewritten as the system of ordinary differential equations

$$\frac{d\mathbf{X}}{dt} = \mathscr{F}_{l,k,\lambda}(\mathbf{X}), \qquad \mathbf{X} = (X_1, ..., X_{18}), \qquad l = 1, \quad k = 2$$

Using the four-step Adams–Bashforth method with step length $h = 0.0002$ and $\lambda = 50$ or 200 again to discretize this set of coupled equations, we find a periodic equation, the phase portraits of which on the (X_8, X_7) plane are displayed in Fig. 5. From this computation it seems valid that Eq. (24) has a unique Hopf bifurcation value and the bifurcated time-dependent periodic solution is stable as λ increases.

6. REMARKS

We have examined steady-state solutions in $H^2_{l,k}$ and time-dependent periodic solutions in $\mathscr{H}^2_{1,2}$ for the Navier–Stokes equations (5)–(6), respectively. However, the subspaces $\tilde{H}^2_{l,k}$ and $\tilde{\mathscr{H}}^2_{1,2}$ are also invariant with respect to Eqs. (5)–(6). It is readily seen that all analysis and computation results on $H^2_{l,k}$ and $\mathscr{H}^2_{l,2}$ replaced respectively by the spaces $\tilde{H}^2_{l,k}$ and $\tilde{\mathscr{H}}^2_{l,2}$ are remain valid.

Thus, when $k = 2$, H^2 has the flow-invariant subspaces $H^2_{l,2}$, $\tilde{H}^2_{l,2}$, $\mathscr{H}^2_{l,2}$, and $\tilde{\mathscr{H}}^2_{l,2}$ for $l \geqslant 0$. The steady-state solution $\psi_0 = -(1/2) \cos 2y$ is always stable for Eqs. (5)–(6) reduced in these subspaces when $l \neq 1$. Pitchfork bifurcation phenomena arise in $H^2_{1,2}$ and $\tilde{H}^2_{1,2}$, respectively, whereas Hopf bifurcation phenomena arise in $\mathscr{H}^2_{1,2}$ and $\tilde{\mathscr{H}}^2_{1,2}$, respectively.

Therefore we can especially give the conclusion for the case $k=2$. Denoting by λ_p the pitchfork bifurcation value and by λ_H the Hopf bifurcation value for Eqs. (5)–(6) with $k=2$, we have $0<\lambda_p<\lambda_H<50$ by computation. When $\lambda<\lambda_p$, the steady-state solution ψ_0 is globally attractive in H^2. When $\lambda_p<\lambda<\lambda_H$, ψ_0 has a one-dimensional unstable manifold in $H^2_{1,2}$ (resp. $\tilde{H}^2_{1,2}$), which now contains a pair of stable steady-state solutions for Eqs. (5)–(6) with $k=2$ reduced in $H^2_{1,2}$ (resp. $\tilde{H}^2_{1,2}$). When $\lambda>\lambda_H$, two time-dependent periodic solutions arise respectively in $\mathscr{H}^2_{1,2}$ and $\tilde{\mathscr{H}}^2_{1,2}$ such that one is stable in $\mathscr{H}^2_{1,2}$ and the other is stable in $\tilde{\mathscr{H}}^2_{1,2}$.

Consequently, any solution of Eqs. (5)–(6) with $k=2$ and $\lambda>0$ starting from either the mode $\cos(mx+ny)$ or the mode $\sin(mx+ny)$ in H^2 is attracted by one of the five steady-state solutions or one of the two periodic solutions. Here m and n are arbitrary integers.

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REFERENCES

1. V. L. Arnold and L. D. Meshalkin, A. N. Kolmogorov's seminar on selected problems of analysis (1958–1959), *Russ. Math. Surv.* **15**(1):247–250 (1960).
2. A. V. Babin and M. L. Vishik, Attractors of partial differential evolution equations and estimates of their dimension, *Russ. Math. Surv.* **38**:151–213 (1983).
3. Z. M. Chen and W. G. Price, Time dependent periodic Navier–Stokes flows on a two-dimensional torus, *Commun. Math. Phys.* **179**:577–597 (1996).
4. P. Constantin, C. Foias, and R. Temam, On the large time Galerkin approximation of the Navier–Stokes equations, *SIAM J. Num. Anal.* **21**:615–634 (1984).
5. V. Franceschini, C. Giberti, and M. Nicolini, Common periodic behavior in larger truncation of the Navier–Stokes equations, *J. Stat. Phys.* **50**:879–896 (1988).
6. D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order* (Springer-Verlag, Berlin, 1977).
7. J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer-Verlag, Berlin, 1983).
8. V. I. Iudovich, Example of the generation of a secondary stationary or periodic flow when there is loss of stability of the laminar flow of a viscous incompressible fluid, *J. Math. Mech.* **29**(3):453–467 (1965).
9. M. S. Jolly, Bifurcation computations on an approximate inertial manifold for the 2-D Navier–Stokes equations, *Physica D* **63**:8–20 (1993).
10. A. Krasnosel'skii, *Topological Methods in the Theory of Non-Linear Integral Equations* (Pergamon, New York, 1964).
11. J. D. Lambert, *Numerical Methods for Ordinary Differential Systems* (Wiley, New York, 1991).
12. V. X. Liu, Instability for the Navier–Stokes equations on the 2-dimensional torus and a lower bound for the Hausdorff dimension of their global attractors, *Commun. Math. Phys.* **147**:217–230 (1992); A sharp lower bound for the Hausdorff dimension of the global attractors 2D Navier–Stokes equations, *Commun. Math. Phys.* **158**:327–339 (1993).

13. V. X. Liu, On unstable and neutral spectrum of incompressible inviscid and viscid fluids on the 2D torus, *Q. Appl. Math.* (1996), to appear.
14. C. Marchioro, An example of absence of turbulence for any Reynolds number, *Commun. Math. Phys.* **105**:99–106 (1986).
15. J. E. Marsden and M. McCracken, *The Hopf Bifurcation and Its Applications* (Springer, Berlin, 1976).
16. M. S. Massey, *A Basic Course in Algebraic Topology* (Springer, Berlin, 1991).
17. L. D. Meshalkin and Ya. G. Sinai, Investigation of the stability of a stationary solution of a system of equations for the plane movement of an incompressible viscous fluid, *J. Math. Mech.* **19**(9):1700–1705 (1961).
18. H. Okamoto and M. Shoji, Bifurcation diagrams in Kolmogorov's problem of viscous incompressible fluid on 2-D flat tori, *Kyoto Univ. Res. Inst. Math. Sci.* **767**:29–51 (1991).
19. R. Temam, *Navier–Stokes Equations* (North-Holland, Amsterdam, 1983).
20. E. S. Titi, On a criterion for locating stable stationary solutions to the Navier–Stokes equations, *Nonlinear Anal. TMA* **11**:1085–1102 (1987).